Applying Precomputation of Integrals to Nonlinear DSGE Models with Occasionally Binding Constraints*

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Abstract

This paper shows that the technique of precomputation of integrals (Judd et al., 2017) can be applied to nonlinear DSGE models with occasionally binding constraints. Specifically, we solve New Keynesian models with the zero lower bound on nominal interest rates with a nonstochastic parameterized expectations algorithm (PEA) augmented with piecewise Chebyshev polynomials and the precomputation technique. We find that the proposed method is significantly faster than, and has accuracy similar to, the methods that compute integrals numerically.

Keywords: Projection methods, Precomputation of integrals, Occasionally binding constraints, Chebyshev PEA.

JEL codes: C63; D52

1 Introduction

The time iteration method (TI) is one of the numerical methods widely used to solve nonlinear dynamic stochastic general equilibrium (DSGE) models.[1] It is known that the standard

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[1]This method is a variant of projection methods with collocation. Coleman (1991) proves the existence of the equilibrium as the fixed point of a functional equation in a stochastic neoclassical growth model with distortionary tax. Greenwood and Huffman (1995) extend it to several cases. See also, e.g., Richter et al. (2014) and Sargent and Stachurski (2018).

TI suffers from costly nonlinear optimization (i.e., root-finding) and numerical integration.[2] By applying the nonstochastic parameterized expectations algorithm (PEA) originally proposed by Christiano and Fisher (2000), we can avoid nonlinear optimization.[3] We can further avoid numerical integration by utilizing the technique of *precomputation of integrals* (Judd et al., 2017). Compared with the standard TI that requires nonlinear optimization and numerical integration, nonstochastic PEAs are more efficient in terms of computation time.

This paper shows that a nonstochastic PEA with the precomputation technique can also be applied to nonlinear DSGE models with occasionally binding constraints. The proposed method is significantly faster than, and has accuracy similar to, the other methods considered in the present paper. For example, it can solve a prototype nonlinear New Keynesian model with the zero lower bound on nominal interest rates within a second in MATLAB without any parallelization, even though the algorithm is also highly parallelizable. Thus, the proposed method would be especially useful when researchers wanted to estimate such a model, as in, e.g., Gust et al. (2017); Iiboshi et al. (2018); Plante et al. (2018).

Marcet (1988) originally develops PEA. It uses a stochastic approach based on Monte Carlo simulations to solve for the coefficients of polynomials that approximate the expectation functions (e.g., the right-hand side of the Euler equation in the stochastic neoclassical growth model).[4] Christiano and Fisher (2000) point out that PEA can be applied to nonstochastic grid points such as Chebyshev collocation points. They use Chebyshev polynomials to approximate the expectation functions and solve for the coefficients of polynomials by a projection method (Judd, 1992). There are two distinct ways to apply nonstochastic PEAs. One is to fit polynomials to future variables (future PEA), and the other is to fit polynomials to current variables (current PEA). In more recent work, Gust et al. (2017) apply the future PEA to solve a nonlinear New Keynesian model with the occasionally binding ZLB constraints.[5]

This paper is based on the previous studies. It proposes a novel approach that solves DSGE models more efficiently. In the proposed method, we use the current PEA so as to precompute integrals of the polynomials analytically. This is a nontrivial task, especially

[2] In the present paper, we use a variant of Newton's method for nonlinear optimization and Gaussian-Hermite quadrature for numerical integration as are often used in solving nonlinear DSGE models.

[3] Endogenous grid point method (Carroll, 2006; Barillas and Fernández-Villaverde, 2007; Fella, 2014) can also avoid nonlinear optimization..

[4] Judd et al. (2011) and Maliar and Maliar (2015) further develop this approach to make the computation more robust and efficient.

[5] They also use Smolyak's method (see, e.g., Judd et al., 2014) for sparse grid points to mitigate the curse of dimensionality, as do we in our numerical example. See, e.g., Hirose and Sunakawa (2019) for a survey of the numerical methods used in this paper.

when we deal with occasionally binding constraints by utilizing an index function for binding constraints (such as in Aruoba et al. 2018; Gust et al. 2017; Hirose and Sunakawa 2015, 2017; Nakata 2017), because an approximated expectation function is now a piecewise polynomial based on two smooth Chebyshev polynomials in which we assume the constraints always or never bind. Under an orthogonality assumption, we can compute the integral of the expectation function as a weighted average of the integrals of the two polynomials with the probability of binding constraints being the weight.[6]

In the reminder of the paper, we introduce a small-scale New Keynesian model with the ZLB in Section 2. In Section 3, we demonstrate how the nonstochastic PEA fitting Chebyshev polynomials to current variables and the precomputation technique can be applied to the New Keynesian model. We solve the model by the proposed method to display numerical results, as well as the nonstochastic PEA fitting future variables (Gust et al., 2017) and the standard TI as a comparison. Section 4 concludes. The details of the standard TI and the future PEA used in Gust et al. (2017) are found in Appendix A.1 and A.2.[7]

2 Small-scale New Keynesian model

The model economy consists of final-good and intermediate-good producing firms, households and monetary and fiscal authorities. Prices are sticky because of Rotemberg-type (1982) adjustment costs. See, e.g., Herbst and Schorfheide (2015); Hirose and Sunakawa (2019) for the details of the model.

[6]The integral of the expectation function itself is *not multiplicatively separable* and the assumption needed for precomputation of expectation functions in Judd et al. (2017) is not satisfied. We need the orthogonality assumption instead as we can precompute only the regime specific functions and the probability of binding constraints.

[7]MATLAB codes are available at `https://github.com/tkksnk/NKZLB/tree/master/smolyak/nkzlb/`.

2.1 Setup

The equilibrium conditions of the New Keynesian model are given by

$$c_t^{-\tau} = \beta\bar{\gamma}^{-1}R_t\mathbb{E}_t\left[\frac{c_{t+1}^{-\tau}}{z_{t+1}\pi_{t+1}}\right], \tag{1}$$

$$\begin{aligned} 0 = &\left((1-\nu^{-1}) + \nu^{-1}c_t^{\tau} - \phi\left(\pi_t - \bar{\pi}\right)\left[\pi_t - \frac{1}{2\nu}\left(\pi_t - \bar{\pi}\right)\right]\right)c_t^{-\tau}y_t \\ &+ \beta\phi\mathbb{E}_t\left[c_{t+1}^{-\tau}y_{t+1}\left(\pi_{t+1} - \bar{\pi}\right)\pi_{t+1}\right], \end{aligned} \tag{2}$$

$$c_t + \frac{\phi}{2}\left(\pi_t - \pi\right)^2 y_t = g_t^{-1}y_t, \tag{3}$$

$$R_t^* = \left(\bar{r}\bar{\pi}\left(\frac{\pi}{\bar{\pi}}\right)^{\psi_1}\left(\frac{y_t}{y_t^*}\right)^{\psi_2}\right)^{1-\rho_R} R_{t-1}^{*\rho_R}e^{\epsilon_{R,t}}, \tag{4}$$

$$R_t = \max\left\{R_t^*, 1\right\}, \tag{5}$$

We have four equations: the consumption Euler equation (1), the New Keynesian Phillips curve (NKPC) (2), the resource constraint (3), and a Taylor-type monetary policy rule (4). c_t is consumption, π_t is the inflation rate, y_t is output, and R_t^* is the notional interest rate that the central bank wishes to set. R_t is the actual nominal interest rate bounded below by one when we consider the ZLB. Government expenditure g_t is exogenous, and the natural level of output is given by $y_t^* = (1-\nu)^{1/\tau}g_t$, which is obtained by setting $\phi = 0$ in the equilibrium conditions. The description of the parameters $(\beta, \tau, \nu, \phi, \bar{r}, \bar{\pi}, \psi_1, \psi_2, \rho_R)$ is given in Table 1 that appears in Section 3.3. The total factor productivity A_t has a deterministic trend $\bar{\gamma}$ and a shock to the trend z_t such as $\ln\gamma_t \equiv \ln(A_t/A_{t-1}) = \ln\bar{\gamma} + \ln z_t$. The exogenous shocks $\{z_t, g_t\}$ follow the AR(1) processes

$$\begin{aligned} \ln z_t &= \rho_z \ln z_{t-1} + \epsilon_{z,t}, \\ \ln g_t &= (1-\rho_g)\ln\bar{g} + \rho_g \ln g_{t-1} + \epsilon_{g,t}, \end{aligned}$$

where ρ_g and ρ_z are the parameters for persistence of the shocks. The disturbance terms $\{\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{R,t}\}$ are serially uncorrelated and independent of each other. The three disturbances are normally distributed with means zero and standard deviations σ_z, σ_g, and σ_R, respectively.

The Coleman operator We define a functional operator on the equilibrium conditions. For the sake of exposition, hereafter we drop time subscripts unless necessary and replace the next period's variables x_{t+1} by x'. The solution to the functional operator takes the form

$$\begin{aligned} c &= \sigma_c(R^*_{-1}, s), & \pi &= \sigma_\pi(R^*_{-1}, s), \\ R^* &= \sigma_{R^*}(R^*_{-1}, s), & y &= \sigma_y(R^*_{-1}, s), \end{aligned}$$

where $s = (z, g, \epsilon_R)$. We have four equilibrium conditions and four endogenous variables to be solved for. Note that the actual policy rate is given by $R = \max\{R^*, 1\}$. The mapping $\sigma = K\sigma$, where $\sigma = [\sigma_c, \sigma_\pi, \sigma_y, \sigma_{R^*}, \sigma_R]'$, solves

$$\begin{aligned}
&0 = -c^{-\tau} + \beta\bar{\gamma}^{-1}R\int\left[\frac{\sigma_c(R^*, s')^{-\tau}}{z'\sigma_\pi(R^*, s')}\right]p(s'|s)ds', \\
&0 = \left((1-\nu^{-1}) + \nu^{-1}c^{\tau} - \phi\left(\pi - \bar{\pi}\right)\left[\pi - \frac{1}{2\nu}\left(\pi - \bar{\pi}\right)\right]\right)c^{-\tau}y \\
&+ \beta\phi\int\left[\sigma_c(R^*, s')^{-\tau}\sigma_y(R^*, s')\left(\sigma_\pi(R^*, s') - \bar{\pi}\right)\sigma_\pi(R^*, s')\right]p(s'|s)ds', \\
&c + \frac{\phi}{2}\left(\pi - \bar{\pi}\right)^2 y = g^{-1}y, \\
&R^* = \left(r\bar{\pi}\left(\frac{\pi}{\bar{\pi}}\right)^{\psi_1}\left(\frac{y}{y^*}\right)^{\psi_2}\right)^{1-\rho_R} {R^*_{-1}}^{\rho_R}e^{\epsilon_R}, \\
&R = \max\{R^*, 1\}
\end{aligned}$$

for σ, where $z' = z^{\rho_z}e^{\epsilon_z'}$, $g'/\bar{g} = (g/\bar{g})^{\rho_g}e^{\epsilon_g'}$, and $p(s'|s)$ is the pdf of s' conditional on s. Note that we substitute $c' = \sigma_c(k', z')$, $\pi' = \sigma_\pi(k', z')$, and $y' = \sigma_y(k', z')$. The operator K takes its argument σ and returns a vector of new functions $K\sigma$ that solves the relevant equations for (c, π, y, R^*, R),

3 Nonstochastic PEA meets precomputation of integral with the ZLB

We focus on the current PEA in the main text and show how the nonstochastic PEA and precomputation technique can be applied to the New Keynesian model. As we compare the standard TI, the future PEA, and the current PEA in a numerical example, we explain the details of the standard TI and the future PEA used in Gust et al. (2017) in Appendix A.1 and A.2.

3.1 Nonstochastic PEA, fitting polynomials to current variables

We define auxiliary expectation functions for the expectation terms in the consumption Euler equation (1) and the NKPC (2) as follows

$$v_c(R^*_{-1}, s) \equiv \beta\bar{\gamma}^{-1}\left[\frac{c^{-\tau}}{z\pi}\right],$$
$$v_\pi(R^*_{-1}, s) \equiv \beta\phi\left[c^{-\tau}y\left(\pi-\bar{\pi}\right)\pi\right].$$

Note that the original expectation terms are obtained as the integral of the expectation function evaluated at the next period's state variables. We adapt an index-function approach as in Aruoba et al. (2018); Gust et al. (2017); Hirose and Sunakawa (2015, 2017); Nakata (2017). For each $x \in \{c, \pi\}$, let $v_{x,\text{NZLB}}(R^*_{-1}, s)$ be the expectation function assuming that ZLB does not bind, and let $v_{x,\text{ZLB}}(R^*_{-1}, s)$ be the expectation function assuming that ZLB binds. Taking a pair of expectation functions $(v_{x,\text{NZLB}}, v_{x,\text{ZLB}})$ for $x \in \{c, \pi\}$ and the policy function for the current notional interest rate in the non-ZLB regime $R^* = \sigma_{R^*,\text{NZLB}}(R^*_{-1}, s)$ as given, we use an index function to obtain

$$v_x(R^*_{-1}, s) = \mathbb{I}_{(R^*<1)}v_{x,\text{ZLB}}(R^*_{-1}, s) + \left(1-\mathbb{I}_{(R^*<1)}\right)v_{x,\text{NZLB}}(R^*_{-1}, s), \tag{6}$$

where the index function depends on the value of R^* and is defined as

$$\mathbb{I}_{(R^*<1)} = \begin{cases} 1 & \text{when } R^* = \sigma_{R^*,\text{NZLB}}(R^*_{-1}, s) < 1, \\ 0 & \text{otherwise.}\end{cases}$$

Time iteration method The modified time iteration method with current PEA takes the following steps:

1. Make an initial guess for the expectation and policy functions $\{v_n^{(0)}, \sigma_n^{(0)}\}$ for $n \in \{\text{NZLB}, \text{ZLB}\}$.

2. Given the expectation and policy functions previously obtained $\{v_n^{(i-1)}, \sigma_n^{(i-1)}\}$, solve the relevant equations for (c, π, y, R^*).

3. Update the expectation and policy functions.

4. Repeat Steps 2–3 until both $\left\|v_n^{(i)} - v_n^{(i-1)}\right\|$ and $\left\|\sigma_n^{(i)} - \sigma_n^{(i-1)}\right\|$ are small enough.

Specifically, in Step 2, taking the piecewise expectation functions $v_x^{(i-1)}(R^*, s')$ for $x \in \{c, \pi\}$ and the values of the policy functions $R^*_{jmn} = \sigma^{(i-1)}_{R^*,n}(R^*_{j,-1}, s_m)$ and $y_{jmn} = \sigma^{(i-1)}_{y,n}(R^*_{j,-1}, s_m)$ for $n \in \{\text{NZLB}, \text{ZLB}\}$ at each grid point indexed by (j, m) as given, we solve

$$
\begin{aligned}
c_{jm\text{NZLB}} &= \left\{ R_{jm\text{NZLB}} \int \hat{v}_c^{(i-1)}(R^*_{jm\text{NZLB}}, s'; \boldsymbol{\theta}) p(s'|s_m) ds' \right\}^{-1/\tau}, \\
0 &= \left((1-\nu^{-1}) + \nu^{-1} c^{\tau}_{jm\text{NZLB}} - \phi\left(\pi_{jm\text{NZLB}} - \bar{\pi}\right) \left[\pi_{jm\text{NZLB}} - \frac{1}{2\nu}\left(\pi_{jm\text{NZLB}} - \bar{\pi}\right) \right] \right) c^{-\tau}_{jm\text{NZLB}} y_{jm\text{NZLB}} \\
&+ \int \hat{v}_\pi^{(i-1)}(R^*_{jm\text{NZLB}}, s'; \boldsymbol{\theta}) p(s'|s_m) ds',
\end{aligned}
$$

for $(c_{jm\text{NZLB}}, \pi_{jm\text{NZLB}})$ and

$$
\begin{aligned}
c_{jm\text{ZLB}} &= \left\{ \int \hat{v}_c^{(i-1)}(R^*_{jm\text{ZLB}}, s'; \boldsymbol{\theta}) p(s'|s_m) ds' \right\}^{-1/\tau}, \\
0 &= \left((1-\nu^{-1}) + \nu^{-1} c^{\tau}_{jm\text{ZLB}} - \phi\left(\pi_{jm\text{ZLB}} - \bar{\pi}\right) \left[\pi_{jm\text{ZLB}} - \frac{1}{2\nu}\left(\pi_{jm\text{ZLB}} - \bar{\pi}\right) \right] \right) c^{-\tau}_{jm\text{ZLB}} y_{jm\text{ZLB}} \\
&+ \int \hat{v}_\pi^{(i-1)}(R^*_{jm\text{ZLB}}, s'; \boldsymbol{\theta}) p(s'|s_m) ds',
\end{aligned}
$$

for $(c_{jm\text{ZLB}}, \pi_{jm\text{ZLB}})$. Note that we interpolate $\hat{v}_x^{(i-1)}(R^*, s'; \boldsymbol{\theta})$ for $x \in \{c, \pi\}$ off the grid points by piecewise Chebyshev polynomials, where $\boldsymbol{\theta}$ is a vector of the coefficients of the polynomials. We use a successive approximation so that we know the values of R^*_{jmn} and y_{jmn}.[8] Then, we analytically obtain c_{jmn} immediately and π_{jmn} as a solution to the second-order polynomial.[9] We show how to compute the integral of the expectation functions in the next section.

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[8] This is also known as fixed-point iteration.

[9] We solve the following second-order polynomial

$$
\alpha_0 - 2\alpha_1 \pi_{jmn} + \alpha_2 \pi^2_{jmn} = 0
$$

for π_{jmn}, where

$$
\begin{aligned}
\alpha_0 =& \frac{\phi \bar{\pi}^2}{2\nu} + (1-\nu^{-1}) + c^{\tau}_{jmn} \left(\nu^{-1} + \frac{\int v_\pi^{(i-1)}(\sigma^{(i-1)}_{R^*,n}(R^*_{j,-1}, s_m), s') p(s'|s_m) ds'}{\sigma^{(i-1)}_{y,n}(R^*_{j,-1}, s_m)} \right), \\
\alpha_1 =& \phi \bar{\pi} \left(\nu^{-1} - 1\right)/2, \\
\alpha_2 =& \phi \left(\frac{1}{2\nu} - 1 \right).
\end{aligned}
$$

We pick up the root $\pi_{jmn} = \alpha_1/\alpha_2 - \sqrt{(\alpha_1/\alpha_2)^2 - \alpha_0}$ of the polynomial and ignore the other root.

In Step 3, we update the expectation and policy functions for each regime $n \in \{\text{NZLB}, \text{ZLB}\}$:

$$\sigma_{y,n}^{(i)}(R_{j,-1}^*, s_m) = y_{jmn} = \left[g_m^{-1} - \frac{\phi}{2}(\pi_{jmn} - \bar{\pi})^2\right]^{-1} c_{jmn},$$

$$\sigma_{R^*,n}^{(i)}(R_{j,-1}^*, s_m) = R_{jmn}^* = \left(\bar{r}\bar{\pi}\left(\frac{\pi_{jmn}}{\bar{\pi}}\right)^{\psi_1}\left(\frac{y_{jmn}}{y^*}\right)^{\psi_2}\right)^{1-\rho_R} R_{j,-1}^{*\rho_R} e^{\epsilon_{R,m}},$$

$$v_{c,n}^{(i)}(R_{j,-1}^*, s_m) = \beta\bar{\gamma}^{-1}\left[\frac{c_{jmn}^{-\tau}}{z_m \pi_{jmn}}\right],$$

$$v_{\pi,n}^{(i)}(R_{j,-1}^*, s_m) = \beta\phi\left[c_{jmn}^{-\tau} y_{jmn}(\pi_{jmn} - \bar{\pi})\pi_{jmn}\right].$$

3.2 Precomputation of integrals with the ZLB

In Step 2 above, we compute the integral of the expectation functions $\hat{v}_x(R^*, s'; \boldsymbol{\theta})$ for $x \in \{c, \pi\}$ with regard to s' for the expectation terms in the consumption Euler equation (1) and the NKPC (2). Note that, as we see in (6), $\hat{v}_x(R^*, s'; \boldsymbol{\theta})$ is a piecewise polynomial based on two smooth polynomials in which we assume that the constraints always or never bind. Then, the integral of the expectation functions can be computed as a weighted average of the integrals of the two polynomials with the probability of binding constraints being the weight. Specifically, we assume that

Assumption. $\mathbb{I}_{(R^{*\prime}<1)}$ *and* $\hat{v}_{x,ZLB}(R^*, s'; \boldsymbol{\theta}) - \hat{v}_{x,NZLB}(R^*, s'; \boldsymbol{\theta})$ *for each* $x \in \{c, \pi\}$ *are orthogonal to each other.*

Then we have

$$\begin{aligned}
&\int \hat{v}_x(R^*, s'; \boldsymbol{\theta}) p(s'|s_m) ds' \\
=&\int \left[\mathbb{I}_{(R^{*\prime}<1)} \hat{v}_{x,\text{ZLB}}(R^*, s'; \boldsymbol{\theta}) + \left(1 - \mathbb{I}_{(R^{*\prime}<1)}\right) \hat{v}_{x,\text{NZLB}}(R^*, s'; \boldsymbol{\theta})\right] p(s'|s_m) ds' \\
=&\int \left[\mathbb{I}_{(R^{*\prime}<1)} \left(\hat{v}_{x,\text{ZLB}}(R^*, s'; \boldsymbol{\theta}) - \hat{v}_{x,\text{NZLB}}(R^*, s'; \boldsymbol{\theta})\right) + \hat{v}_{x,\text{NZLB}}(R^*, s'; \boldsymbol{\theta})\right] p(s'|s_m) ds' \\
=&\Pr(R^{*'} < 1)\left(\int \hat{v}_{x,\text{ZLB}}(R^*, s'; \boldsymbol{\theta}) p(s'|s_m) ds' - \int \hat{v}_{x,\text{NZLB}}(R^*, s'; \boldsymbol{\theta}) p(s'|s_m) ds'\right) \\
&+ \int \hat{v}_{x,\text{NZLB}}(R^*, s'; \boldsymbol{\theta}) p(s'|s_m) ds',
\end{aligned}$$

where $R^{*'} = \sigma_{R^*,\text{NZLB}}(R^*, s')$ is the notional rate in the next period.[10]

[10] If X and Y are orthogonal, $E[XY] = E[X]E[Y]$ holds.

The integral of each $\hat{v}_{x,\text{NZLB}}(R^*, s'; \boldsymbol{\theta})$ and $\hat{v}_{x,\text{ZLB}}(R^*, s'; \boldsymbol{\theta})$ can be precomputed analytically. For the sake of exposition, an ordinary polynomial is used and s includes the technology shock only, $s = z$.[11] The polynomial can be written as

$$\hat{v}_{x,n}(R^*, z'; \boldsymbol{\theta}) = \theta_{x,n,0,0} + \theta_{x,n,1,0}R^* + \theta_{x,n,2,0}(R^*)^2 + \theta_{x,n,0,1}z' + \theta_{x,n,0,2}(z')^2,$$

for each $n \in \{\text{NZLB}, \text{ZLB}\}$. Then, we can analytically obtain the integral as

$$\begin{aligned}
\int \hat{v}_{x,n}(R^*, z'; \boldsymbol{\theta})p(z'|z_m)dz' &= \theta_{x,n,0,0} + \theta_{x,n,1,0}R^* + \theta_{x,n,2,0}(R^*)^2 \\
&+ \int \left(\theta_{x,n,0,1}z' + \theta_{x,n,0,2}(z')^2\right) p(z'|z_m)dz' \\
&= \theta_{x,n,0,0} + \theta_{x,n,1,0}R^* + \theta_{x,n,2,0}(R^*)^2 \\
&+ \int \left(\theta_{x,n,0,1}(\rho z + \epsilon') + \theta_{x,n,0,2}(\rho z + \epsilon')^2\right) p(\epsilon')d\epsilon' \\
&= \theta_{x,n,0,0} + \theta_{x,n,1,0}R^* + \theta_{x,n,2,0}(R^*)^2 + \theta_{x,n,0,1}\rho z + \theta_{x,n,0,2}\rho^2 z^2 + \theta_{x,n,0,2}\sigma_\epsilon^2.
\end{aligned}$$

By contrast, the integral of $\hat{v}_x(R^*, s'; \boldsymbol{\theta}) = \mathbb{I}_{(R^{*\prime}<1)}\hat{v}_{x,\text{ZLB}}(R^*, s'; \boldsymbol{\theta}) + \left(1 - \mathbb{I}_{(R^{*\prime}<1)}\right) \hat{v}_{x,\text{NZLB}}(R^*, s'; \boldsymbol{\theta})$ itself cannot be precomputed either analytically or numerically as the function $\hat{v}_x(R^*, s'; \boldsymbol{\theta})$ is *not multiplicatively separable* in R^* and s' due to the existence of $\mathbb{I}_{(\sigma_{R^*,\text{NZLB}}(R^*,s')<1)}$. Thus the assumption needed for precomputation of expectation functions in Judd et al. (2017) is not satisfied. We need the orthogonality assumption instead as we can precompute only the regime specific functions and the probability of binding constraints.

To compute the probability of binding constraints $\Pr(R^{*'} < 1) = \int \mathbb{I}_{(\sigma_{R^*,\text{NZLB}}(R^*,s')<1)}ds'$, we approximate $\sigma_{R^*,\text{NZLB}}(R^*, s')$ up to the first order by truncating higher-order terms in $\hat{\sigma}_{R^*,\text{NZLB}}(R^*, s'; \boldsymbol{\theta})$

$$\begin{aligned}
\tilde{\hat{\sigma}}_{R^*,\text{NZLB}}(R^*, s'; \boldsymbol{\theta}) =& \theta_0 + \theta_{R^*}R^* + \theta_g g' + \theta_z z' + \theta_r \epsilon_r' \\
=& \theta_0 + \theta_{R^*}R^* + \theta_g \rho g + \theta_z \rho z + \theta_g \epsilon_g' + \theta_z \epsilon_z' + \theta_r \epsilon_r'.
\end{aligned}$$

Then we have

$$\Pr(\tilde{\hat{\sigma}}_{R^*,\text{NZLB}}(R^*, s'; \boldsymbol{\theta}) < 1) = \Phi\left(x < 1 - (\theta_0 + \theta_{R^*}R^* + \theta_g \rho g + \theta_z \rho z)\right),$$

where $\Phi(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}\sigma_x}\right)\right]$ is the cumulative distribution function of $x = \theta_g \epsilon_g' + \theta_z \epsilon_z' +$

[11] A Chebyshev polynomial can be used instead in a straight-forward way.

$\theta_r \epsilon'_r$, which follows a normal distribution with mean zero and variance $\sigma_x^2 = \theta_g^2 \sigma_g^2 + \theta_z^2 \sigma_z^2 + \theta_r^2 \sigma_r^2$.[12]

3.3 Numerical examples

For numerical illustration, the New Keynesian model presented in Section 2 is parameterized according to Table 1. These values except for $(\nu, \bar{g})$ are taken from parameter estimates of the log-linearized version of the model in Herbst and Schorfheide (2015).[13] We use the second- and fourth-order Chebyshev polynomials for R^*_{-1} and $s = (z, g, \epsilon_r)$ for interpolation in each solution algorithm. The numbers of grid points are $3^4 = 81$ and $5^4 = 625$ for each polynomial case. We also use the Smolyak algorithm for each polynomial case. In these cases, the numbers of grid points are 9 and 41 (See Judd et al. 2014; Gust et al. 2017). The number of the Gaussian-Hermite quadrature is set to be $3^3 = 27$.

Table 1: Parameter values of the New Keynesian model.

Parameter		Value
ν	Inverse of demand elasticity	1/6
$\bar{g}$	Steady state government expenditure	1.25
γ	Steady state technology growth	1.0052
β	Discount factor	0.9990
$\bar{\pi}$	Steady state inflation	1.0083
τ	CRRA parameter	2.83
ϕ	Price adjustment cost	17.85
ψ_1	Interest rate elasticity to inflation	1.80
ψ_2	Interest rate elasticity to output gap	0.63
ρ_r	Interest rate smoothing	0.77
ρ_g	Persistence of government shock	0.98
ρ_z	Persistence of technology growth shock	0.88
σ_r	Std. dev. of monetary policy shock	0.0022
σ_g	Std. dev. of government shock	0.0071
σ_z	Std. dev. of technology growth shock	0.0031

We evaluate the accuracy of computation by the residual function errors, which are given

[12] We can possibly use higher-order terms in $\hat{\sigma}_{R^*}$ to calculate the probability at the cost of a more complicated distribution for x.

[13] The parameter for price adjustment cost ϕ is obtained by setting the values of the elasticity of demand $\nu^{-1} = 6$ and the slope of the NKPC (which is a composite function of parameters) $\kappa = \frac{\tau(\nu^{-1}-1)}{\bar{\pi}^2\phi} = 0.78$. The steady-state government expenditure shock is given by $\bar{g} = (1 - g_y)^{-1}$, where g_y is the ratio of government expenditure to output and set at 0.2.

by

$$\mathcal{E}_c(R^*_{-1},s) = 1-\beta\bar{\gamma}^{-1}R\int\left\{\left(\frac{\sigma_c(R^*,s')}{\sigma_c(R^*_{-1},s)}\right)^{-\tau}\frac{1}{z'\sigma_\pi(R^*,s')}\right\}\mu(s'|s)ds' \tag{7}$$

$$\begin{aligned}\mathcal{E}_\pi(R^*_{-1},s) =& (1-\nu^{-1})+\nu^{-1}\sigma_c(R^*_{-1},s)^{-\tau} \\ & -\phi\left(\sigma_\pi(R^*_{-1},s)-\bar{\pi}\right)\left[\sigma_\pi(R^*_{-1},s)-\frac{1}{2\nu}\left(\sigma_\pi(R^*_{-1},s)-\bar{\pi}\right)\right] \\ & +\beta\phi\int\left\{\left(\frac{\sigma_c(R^*,s')}{\sigma_c(R^*_{-1},s)}\right)^{-\tau}\frac{y'}{y}\left(\sigma_\pi(R^*,s')-\bar{\pi}\right)\sigma_\pi(R^*,s')\right\}\mu(s'|s)ds',\end{aligned} \tag{8}$$

where

$$y = \left(g^{-1}-\frac{\phi}{2}\left(\sigma_\pi(R^*_{-1},s)-\bar{\pi}\right)^2\right)^{-1}\sigma_c(R^*_{-1},s), \tag{9}$$

$$R^* = \left(\bar{r}\bar{\pi}\left(\frac{\sigma_\pi(R^*_{-1},s)}{\bar{\pi}}\right)^{\psi_1}\left(\frac{y}{y^*}\right)^{\psi_2}\right)^{1-\rho_R}(R^*_{-1})^{\rho_R}e^{\epsilon_R}. \tag{10}$$

Note that the last two equations of the static equilibrium conditions (9) and (10) hold with equality.s[14] The Euler equation errors are calculated based on the series simulated from the approximated solutions. Each stochastic simulation is done for 10,500 periods, and the first 500 periods are discarded. The same sequence of random variables for s is used throughout all the simulations. Computation is done in MATLAB R2016b using a laptop with Xeon E3-1505Mv5 (2.8Ghz) and 16Gb memory without any parallelization.

Table 2.a shows the results in the case without the ZLB. We compare the performance of the different methods (TI, future PEA, and current PEA) and polynomials (second-order Chebyshev, second-order Smolyak, and fourth-order Smolyak) in terms of the computation time (in seconds), the simulated moments of output growth, inflation, and interest rates, and accuracy measured by the residual function errors in equation (7) and (8). First, we can see that future PEA is faster than TI and that current PEA is faster than future PEA and TI. This is because future PEA avoids only nonlinear optimization and current PEA avoids both numerical optimization and numerical integration. Second, TI, future PEA, and current PEA yield similar second-order simulated moments. The inflation rate and the policy rate are a bit more volatile in future PEA though. Finally, TI, future PEA, and current PEA are also comparable in terms of accuracy.[15]

[14] y' in (8) is calculated by substituting R^* implied by (10) into R^*_{-1} in (9).

[15] The use of Smolyak polynomials makes the computation even faster by the use of sparse grid points.

Table 2.b shows the results in the case with the ZLB. We observe the same pattern in terms of comparison of the computation time, the simulated moments, and accuracy of the different methods and polynomials as in the case without the ZLB. It takes more time to solve the model with the ZLB by the index-function approach. The overall accuracy is a bit worse compared with the case without the ZLB but still reasonable. The probability of binding the ZLB is similar for the different methods, but the methods with second-order Smolyak polynomials have a higher probability of binding the ZLB than others. If we use fourth-order Smolyak polynomials, we can solve the model at an acceptable accuracy within less than a second by using current PEA.

In the rows labeled by cPEA2 in Table 2, we show the results for current PEA without precomputation. We can see that the orthogonality assumption indeed does not matter by comparing the numerical results of the current PEA with precomputation and the current PEA without precomputation but with Gaussian-Hermite numerical integration. The results are almost indistinguishable except for the computation time. As expected, the computation time is longer without precomputation because of the costly numerical integration and is comparable to that for future PEA.

In Table 3 in Appendix A.3, we also check the validity of the orthogonality assumption ex post by calculating the correlation between $\mathbb{I}_{(R^{*\prime}<1)}$ and $\hat{v}_{x,\text{ZLB}}(R^*, s'; \boldsymbol{\theta}) - \hat{v}_{x,\text{NZLB}}(R^*, s'; \boldsymbol{\theta})$ for $x \in \{c, \pi\}$ by Kendall or Spearman rank correlation.[16] The correlation statistics are about 0.1–0.2, implying that the orthogonality assumption is approximately satisfied for the parameter set considered here.

We also check that our results are robust by varying parameter values. We are especially interested in the cases with more frequently binding the ZLB constraints. We consider two cases. One is the case of a low $\bar{\pi}$ and the other is the case with a high σ_z. Specifically, we set $\bar{\pi} = 1.0063$ or $\sigma_z = 0.0037$ so that the probability of binding ZLB increases to about 4% in each case. We solve the models using fourth-order Smolyak polynomials. In Table 4–7, it is found that our results are robust to these cases. That is, current PEA is the fastest and yields similar moments and Euler equation errors as in TI and future PEA. The correlation statistics are slightly higher, about 0.2–0.3.

The use of second-order Smolyak polynomials makes the errors worse compared to second-order Chebyshev polynomials, but fourth-order Smolyak polynomials are comparable with second-order Chebyshev polynomials. Note that the computation time for the methods with fourth-order Smolyak polynomials is still about 5–6 times shorter than the methods with second-order Chebyshev polynomials.

[16] Pearson correlation cannot be used as $\mathbb{I}_{(R^{*\prime}<1)}$ is a binary variable.

4 Conclusion

We have shown that the technique of precomputation of integrals can be applied to nonlinear New Keynesian DSGE models with occasionally binding constraints. The proposed method is a magnitude faster than the other methods considered in the paper, as we avoid both the costly numerical optimization and numerical integration. The method can also be applied to medium- and large-scale New Keynesian models and other nonlinear DSGE models with occasionally binding constraints. The proposed method would be especially useful for researchers who want to estimate these models.

Table 2: Accuracy and speed of TI, future PEA, current PEA.

a. Without ZLB

2nd	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	CPU
TI	-4.15	-2.45	-3.47	-1.79	0.76	1.93	2.36	318.07
fPEA	-4.86	-2.76	-3.52	-2.04	0.76	2.01	2.46	20.90
cPEA	-4.32	-2.46	-3.30	-1.75	0.76	1.92	2.35	1.39
cPEA2	-4.32	-2.46	-3.30	-1.75	0.76	1.92	2.35	23.75

2nd, Smolyak	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	CPU
TI	-3.45	-2.35	-2.34	-1.31	0.76	1.97	2.42	3.71
fPEA	-3.32	-2.66	-2.21	-1.63	0.76	2.13	2.59	0.29
cPEA	-3.36	-2.36	-2.21	-1.34	0.76	1.98	2.43	0.04
cPEA2	-3.36	-2.36	-2.21	-1.34	0.76	1.98	2.43	0.40

4th, Smolyak	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	CPU
TI	-5.09	-3.73	-3.72	-2.57	0.76	1.99	2.43	61.94
fPEA	-5.03	-3.69	-3.71	-2.69	0.76	2.00	2.44	4.06
cPEA	-4.93	-3.75	-3.56	-2.53	0.76	1.99	2.43	0.32
cPEA2	-5.10	-3.01	-3.94	-1.88	0.76	1.99	2.43	5.28

b. With ZLB

2nd	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	$\Pr_{(R^*<1)}$	CPU
TI	-3.73	-2.62	-2.07	-1.42	0.76	2.05	2.53	1.53	1127.3
fPEA	-3.73	-2.67	-2.08	-1.44	0.76	2.11	2.59	1.71	82.28
cPEA	-4.05	-2.71	-2.12	-1.53	0.76	2.01	2.45	1.03	4.05
cPEA2	-3.95	-2.73	-2.16	-1.66	0.76	2.03	2.48	1.25	53.38

2nd, Smolyak	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	$\Pr_{(R^*<1)}$	CPU
TI	-3.40	-2.38	-2.06	-1.09	0.76	2.02	2.50	1.79	12.98
fPEA	-3.26	-2.66	-1.92	-1.48	0.76	2.19	2.68	3.42	0.96
cPEA	-3.35	-2.42	-1.97	-1.25	0.76	2.01	2.47	1.92	0.19
cPEA2	-3.33	-2.43	-1.99	-1.24	0.76	2.03	2.50	2.13	0.92

4th, Smolyak	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	$\Pr_{(R^*<1)}$	CPU
TI	-3.97	-3.14	-2.07	-1.73	0.76	2.04	2.50	1.40	270.65
fPEA	-4.04	-3.54	-2.13	-1.49	0.76	2.03	2.48	1.18	14.66
cPEA	-4.17	-2.95	-2.12	-1.44	0.76	2.03	2.47	1.07	0.99
cPEA2	-4.12	-2.96	-2.13	-1.51	0.76	2.04	2.49	1.29	11.71

Notes: $L_{1,c}$, $L_{1,\pi}$, $L_{\infty,c}$, and $L_{\infty,\pi}$ are, respectively, the average and maximum of Euler errors in absolute values (7)-(8) (in log 10 units) on a 10,000-period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds). $\sigma_{\Delta y}$, σ_{π}, and σ_R are the standard deviations of output growth, inflation, and the policy rate. $\Pr_{(R^*<1)}$ is the probability of the ZLB binding. TI is time iteration, fPEA is future PEA, cPEA is current PEA with precomputation, and cPEA2 is current PEA without precomputation.

References

Aruoba, S. B., P. Cuba-Borda, and F. Schorfheide (2018): "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries," *Review of Economic Studies*, 85, 87–118.

Barillas, F. and J. Fernández-Villaverde (2007): "A generalization of the endogenous grid method," *Journal of Economic Dynamics and Control*, 31, 2698–2712.

Carroll, C. (2006): "The method of endogenous gridpoints for solving dynamic stochastic optimization problems," *Economics Letters*, 91, 312–320.

Christiano, L. and J. Fisher (2000): "Algorithms for solving dynamic models with occasionally binding constraints," *Journal of Economic Dynamics and Control*, 24, 1179–1232.

Coleman, W. J. (1991): "Equilibrium in a Production Economy with an Income Tax," *Econometrica*, 59, 1091–1104.

Fella, G. (2014): "A generalized endogenous grid method for non-smooth and non-concave problems," *Review of Economic Dynamics*, 17, 329–344.

Greenwood, J. and G. W. Huffman (1995): "On the Existence of Nonoptimal Equilibria in Dynamic Stochastic Economies," *Journal of Economic Theory*, 65, 611–623.

Gust, C., E. Herbst, D. López-Salido, and M. E. Smith (2017): "The Empirical Implications of the Interest-Rate Lower Bound," *American Economic Review*, 107, 1971–2006.

Herbst, E. P. and F. Schorfheide (2015): *Bayesian Estimation of DSGE Models*, Princeton University Press.

Hirose, Y. and T. Sunakawa (2015): "Parameter Bias in an Estimated DSGE Model: Does Nonlinearity Matter?" CAMA Working Paper 46/2015.

——— (2017): "The Natural Rate of Interest in a Nonlinear DSGE Model," CAMA Working Paper 38/2017.

——— (2019): "Review of Solution and Estimation Methods for Nonlinear DSGE Models with the Zero Lower Bound," *Japanese Economic Review*, forthcoming.

Iiboshi, H., M. Shintani, and K. Ueda (2018): "Estimating a Nonlinear New Keynesian Model with a Zero Lower Bound for Japan," Tokyo Center for Economic Research Working Paper E-120.

Judd, K. (1992): "Projection methods for solving aggregate growth models," *Journal of Economic Theory*, 58, 410–452.

Judd, K. L., L. Maliar, and S. Maliar (2011): "Numerically stable and accurate stochastic simulation approaches for solving dynamic economic models," *Quantitative Economics*, 2, 173–210.

Judd, K. L., L. Maliar, S. Maliar, and I. Tsener (2017): "How to solve dynamic stochastic models computing expectations just once," *Quantitative Economics*, 8, 851–893.

Judd, K. L., L. Maliar, S. Maliar, and R. Valero (2014): "Smolyak method for solving dynamic economic models: Lagrange interpolation, anisotropic grid and adaptive domain," *Journal of Economic Dynamics and Control*, 44, 92–123.

Maliar, L. and S. Maliar (2015): "Merging simulation and projection approaches to solve high-dimensional problems with an application to a new Keynesian model," *Quantitative Economics*, 6, 1–47.

Marcet, A. (1988): "Solving non-linear models by parameterizing expectations," Carnegie Mellon University, Graduate School of Industrial Administration.

Nakata, T. (2017): "Uncertainty at the Zero Lower Bound," *American Economic Journal: Macroeconomics*, 9, 186–221.

Plante, M., A. W. Richter, and N. A. Throckmorton (2018): "The Zero Lower Bound and Endogenous Uncertainty," *Economic Journal*, 128, 1730–1757.

Richter, A., N. Throckmorton, and T. Walker (2014): "Accuracy, Speed and Robustness of Policy Function Iteration," *Computational Economics*, 44, 445–476.

Rotemberg, J. (1982): "Sticky Prices in the United States," *Journal of Political Economy*, 90, 1187–1211.

Sargent, T. J. and J. Stachurski (2018): *Lectures in Quantitative Economics.*

A Appendix (not for publication)

A.1 Standard time iteration

For each $x \in \{c, \pi, y, R^*\}$, let $\sigma_{x,\text{NZLB}}(R^*_{-1}, s)$ be the policy function assuming that ZLB does not bind, and let $\sigma_{x,\text{ZLB}}(R^*_{-1}, s)$ be the policy function assuming that ZLB binds. Taking a pair of policy functions $\left(\sigma_{x,\text{NZLB}}(R^*_{-1}, s), \sigma_{x,\text{ZLB}}(R^*_{-1}, s)\right)$ as given, we use an index function to obtain

$$\sigma_x(R^*_{-1}, s) = \mathbb{I}_{(R^*<1)}\sigma_{x,\text{ZLB}}(R^*_{-1}, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right)\sigma_{x,\text{NZLB}}(R^*_{-1}, s),$$

where

$$\mathbb{I}_{(R^*<1)} = \begin{cases} 1 & \text{when } R^* = \sigma_{R^*,\text{NZLB}}(R^*_{-1}, s) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The time iteration method takes the following steps.

1. Make an initial guess for the policy function $\sigma^{(0)}$.
2. Taking as given the policy function previously obtained $\sigma^{(i-1)}$, solve the relevant equations for (c, π, y, R^*).
3. Update the policy function by setting $c = \sigma_c^{(i)}(R^*_{-1}, s)$, $\pi = \sigma_\pi^{(i)}(R^*_{-1}, s)$, $R^* = \sigma_\pi^{(i)}(R^*_{-1}, s)$, and $y = \sigma_y^{(i)}(R^*_{-1}, s)$.
4. Repeat Steps 2-3 until $\left\|\sigma^{(i)} - \sigma^{(i-1)}\right\|$ is small enough.

Specifically, in Step 2, taking the policy functions $\sigma_x^{(i-1)}$ for $x \in \{c, \pi, y\}$ as given, we solve

$$c_{jm\text{NZLB}} = \beta\bar{\gamma}^{-1}R_{jm\text{NZLB}} \int \left[\frac{\hat{\sigma}_c^{(i-1)}(R^*_{jm\text{NZLB}}, s'; \boldsymbol{\theta})^{-\tau}}{z'\hat{\sigma}_\pi^{(i-1)}(R^*_{jm\text{NZLB}}, s'; \boldsymbol{\theta})}\right] p(s'|s)ds',$$

$$0 = \left((1 - \nu^{-1}) + \nu^{-1}c^{\tau}_{jm\text{NZLB}} - \phi\left(\pi_{jm\text{NZLB}} - \bar{\pi}\right)\left[\pi_{jm\text{NZLB}} - \frac{1}{2\nu}\left(\pi_{jm\text{NZLB}} - \bar{\pi}\right)\right]\right)c^{-\tau}_{jm\text{NZLB}}y_{jm\text{NZLB}}$$

$$+ \beta\phi \int \left[\hat{\sigma}_c^{(i-1)}(R^*_{jm\text{NZLB}}, s'; \boldsymbol{\theta})^{-\tau}\hat{\sigma}_y^{(i-1)}(R^*_{jm\text{NZLB}}, s'; \boldsymbol{\theta})\right.$$

$$\left.\times\left(\hat{\sigma}_\pi^{(i-1)}(R^*_{jm\text{NZLB}}, s'; \boldsymbol{\theta}) - \bar{\pi}\right)\hat{\sigma}_\pi^{(i-1)}(R^*_{jm\text{NZLB}}, s'; \boldsymbol{\theta})\right] p(s'|s)ds',$$

$$y_{jm\text{NZLB}} = \left[g_m^{-1} - \frac{\phi}{2}\left(\pi_{jm\text{NZLB}} - \bar{\pi}\right)^2\right]^{-1} c_{jm\text{NZLB}},$$

$$R^*_{jm\text{NZLB}} = \left(\bar{r}\bar{\pi}\left(\frac{\pi_{jm\text{NZLB}}}{\bar{\pi}}\right)^{\psi_1}\left(\frac{y_{jm\text{NZLB}}}{y^*}\right)^{\psi_2}\right)^{1-\rho_R} R^{*\rho_R}_{j,-1}e^{\epsilon_{R,m}}.$$

for $(c_{jm\text{NZLB}}, \pi_{jm\text{NZLB}}, y_{jm\text{NZLB}}, R^*_{jm\text{NZLB}})$, and

$$\begin{aligned}
c_{jm\text{ZLB}} &= \beta\bar{\gamma}^{-1} \int \left[\frac{\hat{\sigma}_c^{(i-1)}(R^*_{jm\text{ZLB}}, s'; \boldsymbol{\theta})^{-\tau}}{z'\hat{\sigma}_\pi^{(i-1)}(R^*_{jm\text{ZLB}}, s'; \boldsymbol{\theta})} \right] p(s'|s)ds', \\
0 &= \left((1-\nu^{-1}) + \nu^{-1} c^{\tau}_{jm\text{ZLB}} - \phi\left(\pi_{jm\text{ZLB}} - \bar{\pi}\right) \left[\pi_{jm\text{ZLB}} - \frac{1}{2\nu}\left(\pi_{jm\text{ZLB}} - \bar{\pi}\right) \right] \right) c^{-\tau}_{jm\text{ZLB}} y_{jm\text{ZLB}} \\
&+ \beta\phi \int \left[\hat{\sigma}_c^{(i-1)}(R^*_{jm\text{ZLB}}, s'; \boldsymbol{\theta})^{-\tau} \hat{\sigma}_y^{(i-1)}(R^*_{jm\text{ZLB}}, s'; \boldsymbol{\theta}) \right. \\
&\times \left. \left(\hat{\sigma}_\pi^{(i-1)}(R^*_{jm\text{ZLB}}, s'; \boldsymbol{\theta}) - \bar{\pi} \right) \hat{\sigma}_\pi^{(i-1)}(R^*_{jm\text{ZLB}}, s'; \boldsymbol{\theta}) \right] p(s'|s)ds', \\
y_{jm\text{ZLB}} &= \left[g_m^{-1} - \frac{\phi}{2}\left(\pi_{jm\text{ZLB}} - \bar{\pi}\right)^2 \right]^{-1} c_{jm\text{ZLB}}, \\
R^*_{jm\text{ZLB}} &= \left(\bar{r}\bar{\pi} \left(\frac{\pi_{jm\text{ZLB}}}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y_{jm\text{ZLB}}}{y^*} \right)^{\psi_2} \right)^{1-\rho_R} R^{*\rho_R}_{j,-1} e^{\epsilon_{R,m}}.
\end{aligned}$$

for $(c_{jm\text{ZLB}}, \pi_{jm\text{ZLB}}, y_{jm\text{ZLB}}, R^*_{jm\text{ZLB}})$. Note that we interpolate $\hat{\sigma}_x^{(i-1)}(R^*, s'; \boldsymbol{\theta})$ for $x \in \{c, \pi, y\}$ off the grid points by piecewise Chebyshev polynomials. We also compute numerical integrals with regard to s'.

In Step 3, we update the functions $\sigma_{x,n}^{(i-1)}(R^*_{-1}, s)$ using the results from the previous step.

A.2 Nonstochastic PEA, fitting polynomials to future variables

As in Gust et al. (2017), we define auxiliary functions for the expectation terms in (1) and (2)[17]

$$\begin{aligned}
e_c(R^*_{-1}, s) &\equiv \beta\bar{\gamma}^{-1} R \int \left[\frac{(c')^{-\tau}}{z'\pi'} \right] p(s'|s)ds', \\
e_\pi(R^*_{-1}, s) &\equiv \beta\phi \int \left[\frac{(c')^{-\tau} y'}{y} \left(\pi' - \bar{\pi}\right) \pi' \right] p(s'|s)ds'.
\end{aligned}$$

For each $x \in \{c, \pi\}$, let $e_{x,\text{NZLB}}(R^*_{-1}, s)$ be the expectation function assuming that ZLB does not bind, and let $e_{x,\text{ZLB}}(R^*_{-1}, s)$ be the expectation function assuming that ZLB binds. Taking a pair of expectation functions $(e_{x,\text{NZLB}}, e_{x,\text{ZLB}})$ for $x \in \{c, \pi\}$ and the policy function for the notional rate in the non-ZLB regime $\sigma_{R^*,\text{NZLB}}(R^*_{-1}, s)$ as given, we use an index function to obtain

[17] We include R in $e_c(R^*_{-1}, s)$ and y in $e_\pi(R^*_{-1}, s)$ so that we can avoid solving a root-finding problem.

$$e_c(R_{-1}^*, s) = \mathbb{I}_{(R^*<1)} e_{c,\mathrm{ZLB}}(R_{-1}^*, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right) e_{c,\mathrm{NZLB}}(R_{-1}^*, s),$$
$$e_\pi(R_{-1}^*, s) = \mathbb{I}_{(R^*<1)} e_{\pi,\mathrm{ZLB}}(R_{-1}^*, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right) e_{\pi,\mathrm{NZLB}}(R_{-1}^*, s),$$

where

$$\mathbb{I}_{(R^*<1)} = \begin{cases} 1 & \text{when } R^* = \sigma_{R^*,\mathrm{NZLB}}(R_{-1}^*, s) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The modified time iteration method with future PEA takes the following steps.

1. Make an initial guess for the expectation functions $e_n^{(0)}$ for $n \in \{\mathrm{NZLB}, \mathrm{ZLB}\}$.

2. Taking as given the expectation functions previously obtained $e_n^{(i-1)}$, solve the relevant equations for (c, π, y, R^*) and obtain the policy functions $\sigma^{(i)}$.

3. Update the expectation functions to obtain $e_n^{(i)}$ by interpolating the policy function $\sigma^{(i)}$.

4. Repeat Steps 2-3 until $\left\| e_n^{(i)} - e_n^{(i-1)} \right\|$ is small enough.

In Step 2, taking as given the values of $e_{x,n}^{(i-1)}(R_{j,-1}, s_m)$ for $x \in \{c, \pi\}$ at each grid point indexed by (j, m) and each regime $n \in \{\mathrm{NZLB}, \mathrm{ZLB}\}$, we obtain

$$c_{jmn}^{-\tau} = e_{c,n}^{(i-1)}(R_{j,-1}^*, s_m)$$
$$0 = (1 - \nu^{-1}) + \nu^{-1} c_{jmn}^{\tau} - \phi\left(\pi_{jmn} - \bar{\pi}\right)\left[\pi_{jmn} - \frac{1}{2\nu}\left(\pi_{jmn} - \bar{\pi}\right)\right]$$
$$+ c_{jmn}^{\tau} e_{\pi,n}^{(i-1)}(R_{j,-1}^*, s_m),$$
$$y_{jmn} = \left[g_m^{-1} - \frac{\phi}{2}\left(\pi_{jmn} - \bar{\pi}\right)^2\right]^{-1} c_{jmn},$$
$$R_{jmn}^* = \left(\bar{r}\bar{\pi}\left(\frac{\pi_{jmn}}{\bar{\pi}}\right)^{\psi_1}\left(\frac{y_{jmn}}{y^*}\right)^{\psi_2}\right)^{1-\rho_R} R_{j,-1}^{*\rho_R} e^{\epsilon_{R,m}},$$

for $(c_{jmn}, \pi_{jmn}, y_{jmn}, R_{jmn}^*)$. We analytically obtain c_{jmn} immediately and π_{jmn} as a solution to the second-order polynomial.[18] Then we have the policy functions $\sigma_{c,n}^{(i)}(R_{-1}^*, s)$,

[18] In particular, we solve the following second-order polynomial

$$\alpha_0 - 2\alpha_1 \pi_{jmn} + \alpha_2 \pi_{jmn}^2 = 0$$

$\sigma^{(i)}_{\pi,n}(R^*_{-1}, s)$, and $\sigma^{(i)}_{y,n}(R^*_{-1}, s)$. Note that we do not solve the nonlinear equations by using optimization routines.

In Step 3, we update

$$
\begin{aligned}
e^{(i)}_{c,\mathrm{NZLB}}(R^*_{j,-1}, s_m) &= \beta\bar{\gamma}^{-1} R_{jm\mathrm{NZLB}} \int \left[\frac{\hat{\sigma}^{(i)}_c(R^*_{jm\mathrm{NZLB}}, s'; \boldsymbol{\theta})^{-\tau}}{z'\hat{\sigma}^{(i)}_\pi(R^*_{jm\mathrm{NZLB}}, s'; \boldsymbol{\theta})} \right] p(s'|s_m) ds' \\
e^{(i)}_{\pi,\mathrm{NZLB}}(R^*_{j,-1}, s_m) &= \beta\phi \int \left[\hat{\sigma}^{(i)}_c(R^*_{jm\mathrm{NZLB}}, s'; \boldsymbol{\theta})^{-\tau} \frac{\hat{\sigma}^{(i)}_y(R^*_{jm\mathrm{NZLB}}, s'; \boldsymbol{\theta})}{y_{jm\mathrm{NZLB}}} \right. \\
&\quad \left. \times \left(\hat{\sigma}^{(i)}_\pi(R^*_{jm\mathrm{NZLB}}, s'; \boldsymbol{\theta}) - \bar{\pi} \right) \hat{\sigma}^{(i)}_\pi(R^*_{jm\mathrm{NZLB}}, s'; \boldsymbol{\theta}) \right] p(s'|s_m) ds', \\
e^{(i)}_{c,\mathrm{ZLB}}(R^*_{j,-1}, s_m) &= \beta\bar{\gamma}^{-1} \int \left[\frac{\hat{\sigma}^{(i)}_c(R^*_{jm\mathrm{ZLB}}, s'; \boldsymbol{\theta})^{-\tau}}{z'\hat{\sigma}^{(i)}_\pi(R^*_{jm\mathrm{ZLB}}, s'; \boldsymbol{\theta})} \right] p(s'|s) ds' \\
e^{(i)}_{\pi,\mathrm{ZLB}}(R^*_{j,-1}, s_m) &= \beta\phi \int \left[\hat{\sigma}^{(i)}_c(R^*_{jm\mathrm{ZLB}}, s'; \boldsymbol{\theta})^{-\tau} \frac{\hat{\sigma}^{(i)}_y(R^*_{jm\mathrm{ZLB}}, s'; \boldsymbol{\theta})}{y_{jm\mathrm{ZLB}}} \right. \\
&\quad \left. \times \left(\hat{\sigma}^{(i)}_\pi(R^*_{jm\mathrm{ZLB}}, s'; \boldsymbol{\theta}) - \bar{\pi} \right) \hat{\sigma}^{(i)}_\pi(R^*_{jm\mathrm{ZLB}}, s'; \boldsymbol{\theta}) \right] p(s'|s_m) ds',
\end{aligned}
$$

where the values R^*_{jmn} and y_{jmn} and the policy functions $\hat{\sigma}^{(i)}_x(R^*, s'; \boldsymbol{\theta})$ evaluated at the next period's state (R^*, s') are obtained from the previous step. Note that we interpolate $\hat{\sigma}^{(i)}_x(R^*, s'; \boldsymbol{\theta})$ for $x \in \{c, \pi, y\}$ off the grid points (or equivalently $\hat{e}^{(i-1)}_x(R^*, s'; \boldsymbol{\theta})$ for $x \in \{c, \pi\}$) by piecewise Chebyshev polynomials. We also compute numerical integrals with regard to s'.

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for π_{jmn}, where

$$
\begin{aligned}
\alpha_0 =& \frac{\phi\bar{\pi}^2}{2\nu} + (1 - \nu^{-1}) + e^{(i-1)}_{c,n}(R^*_{j,-1}, s_m)^{-1} \left(\nu^{-1} + e^{(i-1)}_{\pi,n}(R^*_{j,-1}, s_m) \right), \\
\alpha_1 =& \phi\bar{\pi} \left(\nu^{-1} - 1 \right)/2, \\
\alpha_2 =& \phi \left(\frac{1}{2\nu} - 1 \right).
\end{aligned}
$$

We pick up the root $\pi_{jmn} = \alpha_1/\alpha_2 - \sqrt{(\alpha_1/\alpha_2)^2 - \alpha_0}$ of the polynomial and ignore the other root.

A.3 Further numerical results

Table 3: Correlation between $\mathbb{I}_{(R^{*\prime}<1)}$ and $\Delta\hat{v}_{x,n}(R^*, s'; \boldsymbol{\theta})$.

Polynomial	$\text{Corr}(\Delta\hat{v}_{c,n}, \mathbb{I}_{(R^{*\prime}<1)})$		$\text{Corr}(\Delta\hat{v}_{\pi,n}, \mathbb{I}_{(R^{*\prime}<1)})$	
	Kendall	Spearman	Kendall	Spearman
2nd	0.14	0.18	-0.14	-0.18
2nd, Smolyak	0.19	0.24	-0.19	-0.24
4th, Smolyak	0.15	0.18	-0.15	-0.18

Table 4: Accuracy and speed of TI, future PEA, current PEA: Case of low $\bar{\pi}$.

a. Without ZLB

	4th, Smolyak							
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	CPU
TI	-4.77	-3.38	-3.55	-2.29	0.76	2.00	2.44	49.71
fPEA	-4.72	-3.51	-3.49	-2.62	0.76	2.00	2.44	3.88
cPEA	-4.60	-3.35	-3.39	-2.23	0.76	2.00	2.44	0.46
cPEA2	-4.77	-2.98	-3.57	-1.87	0.76	2.00	2.44	5.14

b. With ZLB

	4th, Smolyak								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	$\text{Pr}_{(R^*<1)}$	CPU
TI	-3.69	-2.90	-1.85	-1.82	0.76	2.14	2.63	4.12	257.64
fPEA	-3.80	-3.14	-1.98	-1.41	0.76	2.09	2.58	3.50	13.04
cPEA	-3.96	-2.65	-2.65	-1.40	0.76	2.06	2.51	2.92	1.27
cPEA2	-3.79	-2.69	-1.93	-1.27	0.76	2.14	2.61	3.90	13.68

Notes: $L_{1,c}$, $L_{1,\pi}$, $L_{\infty,c}$, and $L_{\infty,\pi}$ are, respectively, the average and maximum of Euler errors in absolute values (7)-(8) (in log 10 units) on a 10,000-period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds). $\sigma_{\Delta y}$, σ_π, and σ_R are the standard deviations of output growth, inflation, and the policy rate. $\text{Pr}_{(R^*<1)}$ is the probability of the ZLB binding. TI is time iteration, fPEA is future PEA, cPEA is current PEA with precomputation, and cPEA2 is current PEA without precomputation.

Table 5: Correlation between $\mathbb{I}_{(R^{*\prime}<1)}$ and $\Delta\hat{v}_{x,n}(R^*, s'; \boldsymbol{\theta})$: Case of low $\bar{\pi}$.

	$\text{Corr}(\Delta\hat{v}_{c,n}, \mathbb{I}_{(R^{*\prime}<1)})$		$\text{Corr}(\Delta\hat{v}_{\pi,n}, \mathbb{I}_{(R^{*\prime}<1)})$	
	Kendall	Spearman	Kendall	Spearman
4th, Smolyak	0.24	0.29	-0.24	-0.29

Table 6: Accuracy and speed of TI, future PEA, current PEA: Case of high σ_z.

a. Without ZLB

	4th, Smolyak							
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	CPU
TI	-4.98	-3.63	-3.58	-2.44	0.78	2.33	2.90	48.79
fPEA	-4.89	-3.57	-3.54	-2.59	0.78	2.33	2.90	3.52
cPEA	-4.81	-3.63	-3.41	-2.39	0.78	2.33	2.90	0.82
cPEA2	-4.98	-2.94	-3.84	-1.83	0.78	2.33	2.90	4.81

b. With ZLB

	4th, Smolyak								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	$\Pr_{(R^*<1)}$	CPU
TI	-3.54	-2.98	-1.64	-1.77	0.78	2.53	3.15	4.97	301.57
fPEA	-3.65	-3.20	-1.76	-1.32	0.78	2.46	3.07	4.16	11.54
cPEA	-3.97	-2.75	-2.49	-1.28	0.77	2.39	2.96	3.18	1.33
cPEA2	-3.64	-2.74	-1.74	-1.05	0.78	2.53	3.13	4.87	20.45

Notes: $L_{1,c}$, $L_{1,\pi}$, $L_{\infty,c}$, and $L_{\infty,\pi}$ are, respectively, the average and maximum of Euler errors in absolute values (7)-(8) (in log 10 units) on a 10,000-period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds). $\sigma_{\Delta y}$, σ_π, and σ_R are the standard deviations of output growth, inflation, and the policy rate. $\Pr_{(R^*<1)}$ is the probability of the ZLB binding. TI is time iteration, fPEA is future PEA, cPEA is current PEA with precomputation, and cPEA2 is current PEA without precomputation.

Table 7: Correlation between $\mathbb{I}_{(R^{*\prime}<1)}$ and $\Delta\hat{v}_{x,n}(R^*, s'; \boldsymbol{\theta})$: Case of high σ_z.

	$\text{Corr}(\Delta\hat{v}_{c,n}, \mathbb{I}_{(R^{*\prime}<1)})$		$\text{Corr}(\Delta\hat{v}_{\pi,n}, \mathbb{I}_{(R^{*\prime}<1)})$	
	Kendall	Spearman	Kendall	Spearman
4th, Smolyak	0.25	0.30	-0.25	-0.30