Applying the Explicit Aggregation Algorithm to Heterogeneous Agent Models in Continuous Time*

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Abstract

This paper applies the explicit aggregation (XPA) algorithm to the standard heterogeneous agent model with aggregate uncertainty in continuous time. We find that the XPA algorithm is faster in solving the model than the Krusell–Smith algorithm, because the XPA algorithm does not rely on simulations to solve the model. The XPA algorithm is more accurate than the perturbation method when aggregate uncertainty is large.

Keywords: Continuous Time, Heterogeneous Agent Models, Explicit Aggregation Algorithm.
JEL codes: C63; D52

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1 Introduction

There is more interest in heterogeneous agent macro models than ever before. Recent studies such as Achdou et al. (2017) and Ahn et al. (2018) apply newly developed numerical methods to solve heterogeneous agent models in continuous time. In particular, Ahn et al. (2018) study heterogeneous agent models with aggregate uncertainty in continuous time and are able to solve the model quickly. However, there are some challenges to their approach. First, their method solves the model using a linear approximation and thus does not capture nonlinear effects of aggregate uncertainty. Second, because of the linearization, the accuracy of solving the model is significantly compromised when aggregate uncertainty is large and/or the model itself is highly nonlinear.

We present an alternative numerical method to address the issues mentioned above. We introduce the explicit aggregation (XPA) algorithm of den Haan and Rendahl (2010) into the standard heterogeneous agent model with aggregate shocks of Krusell and Smith (1998) in continuous time. Then we compare the XPA algorithm in terms of accuracy and efficiency with the Krusell–Smith (KS) algorithm using simulations and the Reiter–Ahn (REITER) algorithm using perturbations around the deterministic steady state.[1]

We find that, compared with the KS algorithm, the XPA algorithm is faster than and as accurate as the KS algorithm in solving the standard Krusell–Smith model. Compared with the REITER algorithm, the XPA algorithm can solve the model nearly as quickly as the REITER algorithm does, and is more accurate than the REITER algorithm, especially when aggregate uncertainty is large.

Our study is closely related to at least two areas of research. One is the literature on the XPA algorithm. den Haan and Rendahl (2010) is the first paper to apply this method to the standard heterogeneous agent model in Krusell and Smith (1998). Sunakawa (2020) applies their approach to some other heterogeneous agent models such as Khan and Thomas (2003, 2008) and Krueger et al. (2016) in discrete time. To the best of our knowledge, the present paper is the first to apply the XPA algorithm to the standard heterogeneous agent model with aggregate uncertainty in continuous time.

[1] Although we focus on these algorithms, some other useful algorithms are also found in den Haan et al. (2010).

The other is the research on methods to solve heterogeneous agent models with aggregate shocks in continuous time. The pioneering research in this area is Ahn et al. (2018). They adapt the perturbation method originally developed by Reiter (2009) to heterogeneous agent models with aggregate shocks in continuous time.[2] Fernández-Villaverde et al. (2019) propose a neural-network algorithm to solve heterogeneous agent models with aggregate shocks in continuous time. Our algorithm, unlike Fernández-Villaverde et al. (2019) and the standard KS algorithm, solves the model without using simulations. Furthermore, whereas the REITER algorithm solves the model using a linear approximation, our algorithm solves the model nonlinearly so as to capture nonlinear effects of aggregate uncertainty.[3] Furthermore, as we will discuss later, the accuracy of solving the model is much higher than what is reported in Ahn et al. (2018), especially when aggregate uncertainty is large. Our results also hold for different degrees of persistence of the aggregate shock.

The paper consists of the following sections. In Section 2, we apply the XPA algorithm, as well as the KS and REITER algorithms, to the Krusell and Smith (1998) model in continuous time. In Section 3, we compare the results of the three algorithms, XPA, KS, and REITER, in terms of accuracy and efficiency. Finally, Section 4 concludes.

2 Algorithms

We apply the XPA algorithm first developed by den Haan and Rendahl (2010) to the Krusell and Smith (1998) model in continuous time as studied by Ahn et al. (2018). We choose the Krusell–Smith model as it is known as one of the most popular heterogeneous agent models with aggregate uncertainty. Applying the XPA algorithm to other models is also straightforward as shown in Sunakawa (2020). In the model, there are a representative firm, and a government, and heterogeneous households whose asset holdings and productivity are different from each other. As the model is well known, we defer the details of the model to the Appendix A.

[2] Reiter (2010a); Ahn et al. (2018) further develop a method to reduce the dimension of the state space by projecting the distribution onto principal components. Bayer and Luetticke (2020) and Childers (2018) also suggest novel approaches using linearization.

[3] Reiter's (2010b) backward induction method can also be applied to solve heterogeneous agent models nonlinearly. The method is applied to the stochastic overlapping generations model with aggregate uncertainty of Khan (2017); Kim (2018). Okahata (2018) demonstrates that the method can also be merged with the continuous-time methods in Ahn et al. (2018).

The XPA algorithm assumes *approximate aggregation* (Young, 2005) as the KS algorithm assumes so that the wealth distribution is approximated by the mean. In contrast with the KS algorithm, the XPA algorithm calculates the forecasting rules without simulations.

Both the XPA and KS algorithms require two types of calculations, *the inner loop and the outer loop*. The inner loop calculation is common between the XPA and KS algorithms.[4] In continuous time models, the finite difference method is used to solve the Hamilton–Jacobi–Bellman (HJB) equation as in Achdou et al. (2017). Given the forecasting rule $\dot{K} = \Gamma(K, Z)$, we solve the HJB equation

$$\begin{aligned}\rho v(a,z,K,Z) = \max_c u(c) &+ v_a(a,z,K,Z)\dot{a} \\ &+ \lambda_z(v(a,z',K,Z) - v(a,z,K,Z)) + v_K(a,z,K,Z)\dot{K} \\ &+ v_z(a,z,K,Z)(-\mu Z) + \frac{\sigma^2}{2}v_{zz}(a,z,K,Z) \end{aligned} \tag{1}$$

for the policy function of household savings, $s(a, z, K, Z)$.

In the outer loop, the policy function $s(a, z, K, Z)$ obtained in the inner loop is used to obtain the forecasting rule for the next period's aggregate capital as follows

$$\begin{aligned}\dot{K} &= \Gamma(K,Z) \\ &= \sum_z \int_a s(a,z;K,Z)g(a,z)da. \end{aligned} \tag{2}$$

The XPA and KS algorithms are different in how they represent Equation (2). In the XPA algorithm, the forecasting rule is

$$\Gamma_{\text{XPA}}(K,Z) = \sum_z \{s(K(z),z,K,Z) + \xi(z)\}\phi(z),$$

where $K(z)$ is capital conditioned on labor productivity, z, $\phi(z) = \int g(a,z)da$ is the proportion of

[4] Fernández-Villaverde et al. (2019) describes the KS algorithm in continuous time in detail. In particular, they solve the standard Krusell–Smith model in continuous time using the KS algorithm.

households with z, and $\xi(z)$ is for correcting the biases due to Jensen's inequality. These objects can be calculated immediately when we compute the steady state. That is, to obtain the forecasting rule, we just need to evaluate the policy function at $a = K(z)$. The value of $K(z)$ may not be on the grid of a, so we use linear interpolation. In Appendix B, we explain the details of the XPA algorithm following den Haan and Rendahl (2010) and Sunakawa (2020).

In the KS algorithm, we simulate the model to obtain the total factor productivity (TFP) sequence Z_t and the mean of the wealth distribution K_t at each point in time. Then the forecasting rule is obtained by estimating the following forecasting rule using the simulated sequence of $\{K_t, Z_t\}$.

$$\Gamma_{\mathrm{KS}}(K, Z) = \beta_0 + \beta_1 \ln K + \beta_2 \ln Z,$$

where $(\beta_0, \beta_1, \beta_2)$ are coefficients estimated by ordinary least squares.

The REITER algorithm also differs from the XPA and KS algorithms in that the approximate aggregation does not hold. Instead, the REITER algorithm linearizes the model around the deterministic steady state and uses the system of linearized equations to solve for the dynamics of the economy in the event of aggregate shocks. See Ahn et al. (2018) for more details.

3 Numerical results

We compare the numerical results from three algorithms; XPA algorithm, KS algorithm, and REITER algorithm. First, we illustrate the property of the forecasting rules obtained by XPA and KS. Then, we demonstrate the simulation results for XPA, KS, and REITER. After that, we discuss the computation time for each algorithm and the accuracy using the Den haan Error proposed by den Haan (2010).[5]

Benchmark parameters

We use the parameters in Table 1 in the benchmark case, following those used in Ahn et al. (2018). Later, we change the volatility and persistence parameters of the TFP to examine its effect on the

[5]We use the code used in Fernández-Villaverde et al. (2019) for KS and the code used in Ahn et al. (2018) for REITER. Both of the codes are modified to use the same set of parameters simply for the purpose of comparison.

accuracy of solving the model between different algorithms.

Table 1: Parameter values

Parameters	Benchmark Value
γ : Relative Risk Aversion	1.0
ρ : Rate of Time Preference	0.01
α : Capital Share	0.36
δ : Rate of Capital Depreciation	0.025
τ : Tax Rate of Labor Income	0.011
b : Rate of Compensation	0.15
μ : Persistence of TFP	0.25
σ : Volatility of TFP	0.07
z : Idiosyncratic of Labor Productivity	$z_u = 0, z_e = 1$
λ : Probability of Labor Productivity	$\lambda_e = 0.50, \lambda_u = 0.03$

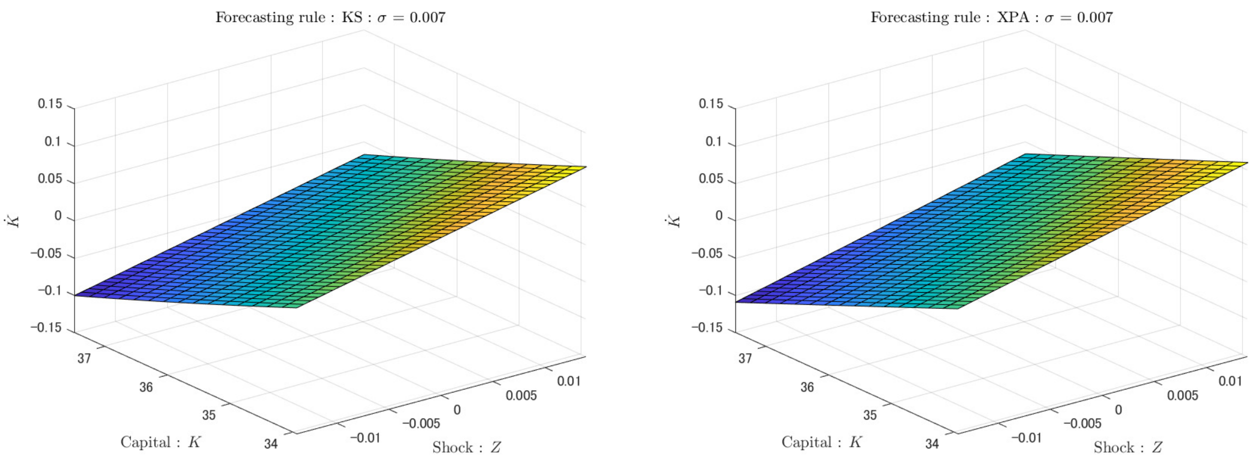
3.1 Forecasting rules

In Figure 1, we show the forecasting rule $\dot{K}(K, Z)$ in KS and XPA. As is clear from the figure, there is no significant difference in the forecasting rules obtained by each algorithm. Each forecasting rule is characterized by a decreasing function with respect to capital K and an increasing function with respect to TFP Z. Furthermore, the slope of the forecasting rule with regard to K is flatter than the 45-degree line as is the case in the standard neoclassical growth model. Thus, with respect to capital, if there is more (less) capital in the current period than in the steady state, $\dot{K}$ is negative (positive) and households will expect the future capital stock to decrease (increase). Furthermore, if TFP is high (low), $\dot{K}$ is positive (negative) and households expect their capital stock to increase (decrease).

3.2 Simulation paths

Next, we compare the capital paths obtained from the simulation results in KS, XPA, and REITER. In the left panel of Figure 2, we show the results of the simulation path derived from the full model (i.e., the forecasting rule and household HJB equation) for 10000 periods in KS, XPA, and REITER. The blue line in the figure shows the capital path by XPA, the red line shows the capital path by KS, and the black line indicates the path by REITER. It is clear from the figure that the capital

Figure 1: Forecasting rules



paths obtained from each algorithm are very similar, except for REITER when capital is low.

XPA and KS solve the model nonlinearly, whereas REITER solves the model by linear approximation. As the law of motion for aggregate capital is concave, the error between the simulation results of the nonlinear methods (XPA and KS) and the linear method (REITER) is small when capital is high. However, when capital is low, the error between the simulation results of the nonlinear and linear methods is large. Thus, in the right panel of Figure 2, we show that the simulation path of REITER diverges from those of XPA and KS as the value of capital decreases.

3.3 Efficiency

Table 2 summarizes the time that it takes to solve the model with each algorithm. Comparing the results of XPA and KS, we can see that XPA solves the model much faster than KS because it does not use simulations. Comparing the computation times of XPA and REITER, we can see that REITER is slightly faster than XPA. However, the difference between the two is not that large (XPA: 2.4 seconds vs. REITER: 0.5 seconds).[6]

[6] Note that we use MATLAB and no parallelization, so the gap might be smaller when we use a faster language and/or parallelization.

Figure 2: Simulation paths

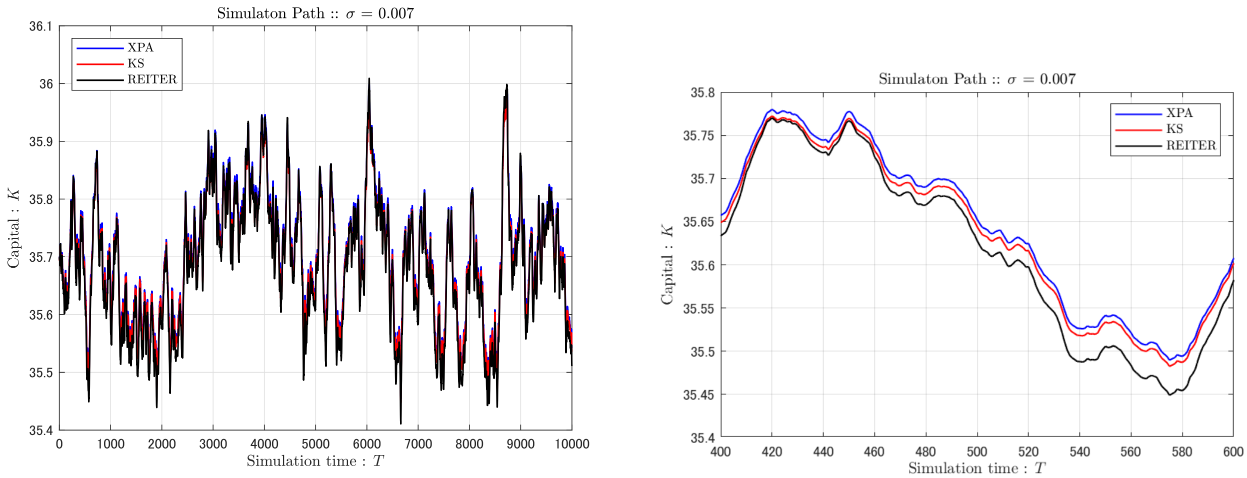


Table 2: Computation time

Algorithm	Computation time
XPA (Our method)	2.478 sec
Krusell-Smith	122.101 sec
REITER (Ahn et al., 2018)	0.501 sec

Notes: Computations are done on a laptop with Intel Core i7-9750H and 16GB RAM using MATLAB R2019a.

3.4 Accuracy

The accuracy of solving the model is discussed using the Den haan errors proposed by den Haan (2010). If the Den haan errors are large in one algorithm, the model solution using this method is not accurate because households are acting based on an erroneous forecasting rule. In this study, we simulate 10000 time periods with each algorithm and use the results from $\{K_t^*\}_{t\in[0,T]}$ and $\{\tilde{K}_t\}_{t\in[0,T]}$ to measure the Den haan errors

$$\varepsilon_{DH}^{MAX} \equiv 100 \cdot \max_{t\in[0,T]} |\ln \tilde{K}_t - \ln K_t^*|,$$

$$\varepsilon_{DH}^{MEAN} \equiv 100 \cdot \frac{\sum_{t\in[0,T]} |\ln \tilde{K}_t - \ln K_t^*|}{T}.$$

Table 3 and Figure 3 summarize the Den haan errors for each algorithm. It is clear from the table that when aggregate uncertainty is small, i.e., when σ is low, there is not much difference in the Den haan errors for each algorithm. However, when aggregate uncertainty is large, i.e., when σ is large, the Den haan errors for REITER are considerably larger than those for XPA and KS. Therefore, it is clear that XPA is able to compute the model more accurately than REITER when aggregate uncertainty is large. Furthermore, in the benchmark case, the Den Haan errors of XPA are smaller than those of KS when the volatility of TFP is $\sigma = 5.0\%$.[7]

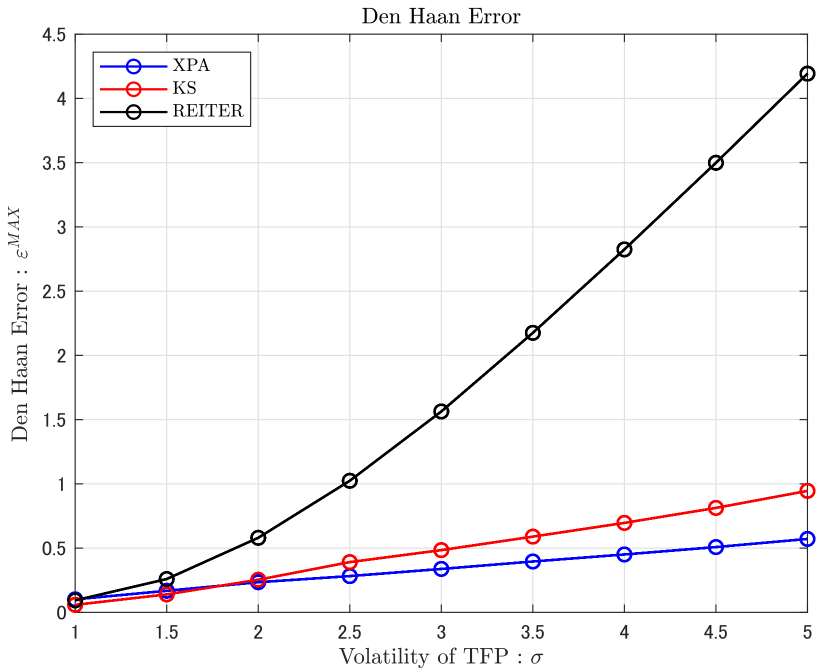
Table 3: Den Haan errors

Agg Shock σ	0.01%	0.1%	0.7%	1.0%	5.0%
$\varepsilon_{DH\ XPA}^{MAX}$	0.000%	0.009%	0.071%	0.101%	0.571%
$\varepsilon_{DH\ XPA}^{MEAN}$	0.000%	0.002%	0.016%	0.024%	0.136%
$\varepsilon_{DH\ KS}^{MAX}$	0.000%	0.004%	0.035%	0.058%	0.945%
$\varepsilon_{DH\ KS}^{MEAN}$	0.000%	0.003%	0.023%	0.037%	0.690%
$\varepsilon_{DH\ REITER}^{MAX}$	0.000%	0.001%	0.044%	0.093%	4.193%
$\varepsilon_{DH\ REITER}^{MEAN}$	0.000%	0.001%	0.038%	0.078%	3.477%

Notes: For each value of σ, we adjust the values of the grid so that the maximum and minimum values of the capital stock obtained in the simulation are within the range of the grid of the capital stock.

[7] We also confirm that our results hold when the persistence of TFP, $1 - \eta$, is lowered so that $\eta = 0.5$ or $\eta = 0.75$. See Appendix C.

Figure 3: Comparison of Den Haan errors



4 Conclusion

In this paper, we apply the explicit aggregation (XPA) algorithm proposed by den Haan and Rendahl (2010) to the standard heterogeneous agent model with aggregate uncertainty (Krusell and Smith, 1998) in continuous time.

We find that, compared with the popular REITER algorithm for solving the heterogeneous agent model with aggregate uncertainty in continuous time, the XPA algorithm is able to solve the model almost as fast as the REITER algorithm, and more accurately than this algorithm in the case when aggregate uncertainty is large.

In our future research, we will apply our algorithm to the heterogeneous agent New Keynesian model (e.g., Bayer et al., 2019; Gornemann et al., 2016; Kaplan et al., 2018) in continuous time with a zero lower bound (ZLB) on nominal interest rates as the XPA algorithm can deal with uncertainty stemming from future exogenous shocks and ZLB (Nakata, 2017). Furthermore, Terry

(2017) shows that the model can be solved more accurately by using a projection method like our method, rather than a perturbation method, when there is a dependency between aggregate uncertainty and idiosyncratic risk. Our algorithm can be applied when this is the case. In these applications, our algorithm will be even more interesting and useful than other algorithms.

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Appendix (not for publication)

A The Krusell–Smith model in continuous time

A.1 Environments

Households

Households face idiosyncratic uncertainty regarding labor productivity and the borrowing constraint $a_t \geq 0$. There are two states of labor productivity for each household, z_e and z_u, which follow the Poisson process with arrival rates λ_e and λ_u. z_e shows that the household is employed and z_u indicates that the household is unemployed.

If the household is employed, she/he receives labor income after taxation $(1-\tau)w_t$. When the household is unemployed, she/he receives unemployment insurance bw_t financed by the labor income tax.

Each household chooses their consumption in each period to maximize their expected life-time utility by taking the wage rate w_t and the interest rate r_t as given.

$$
\begin{aligned}
& v = \max \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \right], \\
\text{s.t.} \quad & da_t = (r_t a_t + (1-\tau) z_t w_t + (1-z_t) b w_t - c_t) dt, \ a_t \geq 0, \\
& z_t \in \{z_e, z_u\}, \ z_e = 1, \ z_u = 0.
\end{aligned}
$$

The instantaneous utility function is of constant-relative-risk-aversion form, ρ is the rate of time preference, and θ is the degree of relative risk aversion.

Firm

The representative firm produces the final good Y_t using capital K_t and labor L_t. The production function is Cobb–Douglas form

$$
Y_t = e^{Z_t} K_t^{\alpha} N_t^{1-\alpha},
$$

where α is the capital share. Z_t is the logarithm of total factor productivity (TFP) following the Ornstein–Uhlenbeck process

$$dZ_t = \eta(\bar{Z} - Z_t)dt + \sigma dW_t, \quad \bar{Z} = 0,$$

where dW_t follows a Wiener process. η is the persistence of TFP and σ is the volatility of the TFP. This process is similar to an AR(1) process in discrete time.

The wage rate and the interest rate are obtained from the first-order conditions for the profit maximization problem as follows:

$$w_t = (1-\alpha)e^{Z_t}K_t^{\alpha}N_t^{-\alpha}, \;\; r_t = \alpha e^{Z_t}K_t^{\alpha-1}N_t^{1-\alpha} - \delta,$$

where δ is the depreciated rate of capital.

Government

The government imposes a tax on labor income to finance unemployment compensation. The government's budget is balanced as below

$$\tau w \mu_e = b w \mu_u.$$

That is, the government's tax revenue from labor income is equal to the government's expenditure to finance unemployment insurance. $\mu_e = \phi(z_e)$ is the share of employment and $\mu_u = \phi(z_u)$ is the share of unemployment in the economy.

A.2 Stationary equilibrium without aggregate uncertainty

We define the steady state without aggregate uncertainty by setting $Z_t = 0$ for all $t \geq 0$. In the steady state without aggregate uncertainty, the following equations are satisfied:

- The Hamilton–Jacobi–Bellman (HJB) equation and the policy function for households

$$\rho v(a,z) = \max_c u(c) + v_a(a,z)(ra + (1-\tau)zw + (1-z)bw - c) + \lambda_z(v(a,z') - v(a,z))$$

$$s(a,z) = ra + (1-\tau)zw + (1-z)bw - c(a,z)$$

- The Fokker–Planck equation

$$0 = \frac{\partial(s(a,z)g(a,z))}{\partial a} - \lambda_z g(a,z) + \lambda_{z'} g(a,z')$$

- The wage rate and the interest rate

$$w = (1-\alpha)K^{\alpha}L^{-\alpha},\ r = \alpha K^{\alpha-1}L^{1-\alpha} - \delta$$

- The government's budget constraint

$$\tau w \mu_e = bw\mu_u,\ \mu_e = \int g(a,z_e)da,\ \mu_u = \int g(a,z_u)da,$$

- The capital and the labor markets clear

$$K = \sum_z \int ag(a,z)da,\ L = \sum_z \int zg(a,z)da$$

B Details of the XPA algorithm

We provide details of the XPA algorithm following den Haan and Rendahl (2010) and Sunakawa (2020). First, we rewrite the wealth distribution $g(a,z)$ using the conditional probability as

$$\begin{aligned} g(a|z) &= \frac{g(a,z)}{\int g(a,z)da} = \frac{g(a,z)}{\phi(z)} \\ \Leftrightarrow g(a,z) &= g(a|z)\phi(z) \end{aligned}$$

where $\phi(z) = \int g(a,z)da$ is equal to the proportion of households with labor productivity z in the economy.[8] Then we can rewrite the forecasting rule using the conditional distribution of wealth as

$$\begin{aligned}\dot{K}(K,Z) &= \sum \int s(a,z;K,Z)g(a,z)da \\ &\approx \sum s\left(\int ag(a|z)da, z; K, Z\right)\phi(z) \\ &= \sum s(K(z),z;K,Z)\phi(z)\end{aligned}$$

where $K(z) = \int ag(a|z)da$ is capital conditioned on labor productivity z. We compute $K(z)$ by the following equations:

$$\begin{aligned}K(z) &= \psi(z)K_{ss}, \;\; \psi(z) \equiv \frac{K_{ss}(z)}{K_{ss}} = \frac{K_{ss}(z)}{\sum_z \int ag_{ss}(a,z)da}, \\ K_{ss}(z) &= \int ag_{ss}(a|z)da = \frac{\int ag_{ss}(a,z)da}{\phi(z)},\end{aligned}$$

where K_{ss} and g_{ss} are capital and wealth distribution at the steady state without aggregate uncertainty. Note that the ratio of the capital conditioned on z to aggregate capital, $\psi(z) = K_{ss}(z)/K_{ss}$, can be easily obtained in the steady-state calculation.[9]

Moreover, following den Haan and Rendahl (2010), we conduct bias correction. In the explicit aggregation, we *assume* that the household's policy function $s(a,z;K,Z)$ is linear at $a = K(z)$ so that $\int s(a,z;K,Z)g(a|z)da \approx s\left(K(z),z;K,Z\right)$ holds. Therefore, there may be bias in the forecasting rule from Jensen's inequality. We compute the steady-state counterparts to correct the bias:

$$\begin{aligned}\xi(z) &= \dot{K}_{ss}(z) - s_{ss}(K_{ss}(z),z), \\ \dot{K}_{ss}(z) &= \int s_{ss}(a,z)g_{ss}(a|z)da = \frac{\int s_{ss}(a,z)g_{ss}(a,z)da}{\phi(z)}.\end{aligned}$$

Again, $\xi(z)$ can be computed at negligible cost in the steady state. Finally, we can write the

[8] We assume the share of employment $\phi(z_e)$ $(= 1 - \phi(z_u))$ is time-invariant, although it is straightforward to make the employment measure be time-variant and depend on aggregate uncertainty as in Krusell and Smith (1998).

[9] We assume that $\psi(z)$ is constant even with aggregate uncertainty. The details of the steady-state calculations are found in Appendix A.2.

forecasting rule as follows

$$\dot{K}(K,Z) = \sum_{z}\{s(K(z),z,K,Z)+\xi(z)\}\phi(z).$$

That is, to obtain the forecasting rule, we just need to evaluate the policy function at $a = K(z)$. The value of $K(z)$ may not be on the grid of a, so we use linear interpolation.

Summary of XPA Algorithm

In summary, we perform computations to solve the Krusell–Smith model with the XPA algorithm as follows. As mentioned above, the XPA algorithm is fast because it does not use any simulations.

1. Compute the deterministic steady state without aggregate uncertainty to obtain the conditional capital ratio $\psi(z)$ and the bias correction term for correcting the forecasting rule $\xi(z)$.

2. (Inner loop) Solve the HJB equation for the policy function taking the forecasting rule as given.

3. (Outer loop) Compute the forecasting rule without simulations taking the policy function as given. The bias correction is also done.

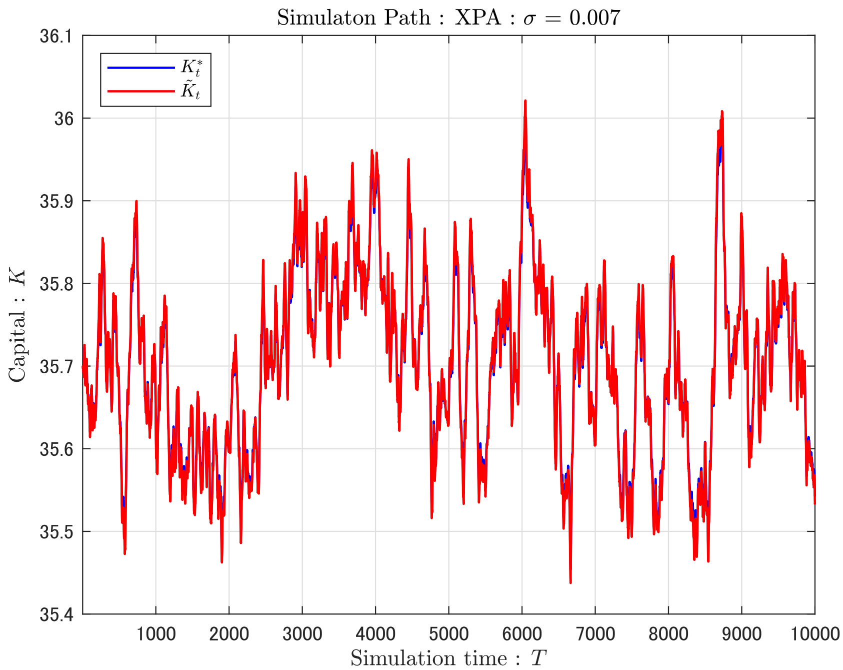
4. Repeat steps 2–3 until the forecasting rule converges.

C Further numerical results

In Figure 4, we show the results of the simulation path in XPA. The red line $\{\tilde{K}_t\}_{t\in[0,T]}$ in the figure shows the simulation obtained only from the forecasting rule for 10000 periods, while the blue line $\{K_t^*\}_{t\in[0,T]}$ shows the simulation obtained from the full model including the forecasting rule and the household HJB equation for 10000 periods.[10] This is known as the Den haan's (2010)

[10] The time interval $dt = 0.25$ used in the simulation is the same as the time interval of Ahn et al. (2018). We use linear interpolation for aggregate capital and TFP to calculate the wealth distribution at each point of time.

Figure 4: Simulation path in XPA (Den haan's fundamental plot)



fundamental plot showing the accuracy of the solution. It is clear from the figure that the capital paths resulting from these simulations are very close.[11]

We also check the robustness of our results with respect to the persistence of TFP. In Table 4, we show that the Den haan errors of KS, XPA, and REITER when the persistence of TFP, $1-\eta$, is lowered so that $\eta = 0.5$ or $\eta = 0.75$. It is clear that, regardless of the persistence, the Den haan errors of REITER are larger than those of XPA and KS when aggregate uncertainty is large.[12]

[11] If the red and blue lines are close, households approximately act according tothe correct forecasting rule. If not, households are using the wrong forecasting rule. Therefore, the divergence between the two lines indicates that the model is not solved correctly based on rational expectations.

[12] However, unlike in the benchmark case of $\eta = 0.25$, the Den Haan errors of XPA are larger than that of KS when the volatility of TFP is $\sigma = 5\%$ and $\eta = 0.5$ or $\eta = 0.75$.

Table 4: Den Haan errors: robustness

a. Case of $\eta = 0.50$

Agg Shock σ	0.01%	0.1%	0.7%	1.0%	3.0%	5.0%
$\varepsilon^{MAX}_{DH\ XPA}$	0.002%	0.019%	0.143%	0.198%	0.607%	1.051%
$\varepsilon^{MEAN}_{DH\ XPA}$	0.000%	0.005%	0.039%	0.054%	0.163%	0.301%
$\varepsilon^{MAX}_{DH\ KS}$	0.000%	0.002%	0.021%	0.034%	0.220%	0.432%
$\varepsilon^{MEAN}_{DH\ KS}$	0.000%	0.001%	0.007%	0.010%	0.049%	0.185%
$\varepsilon^{MAX}_{DH\ REITER}$	0.000%	0.001%	0.067%	0.150%	2.387%	5.862%
$\varepsilon^{MEAN}_{DH\ REITER}$	0.000%	0.001%	0.055%	0.121%	1.953%	4.778%

b. Case of $\eta = 0.75$

Agg Shock σ	0.01%	0.1%	0.7%	1.0%	3.0%	5.0%
$\varepsilon^{MAX}_{DH\ XPA}$	0.003%	0.038%	0.276%	0.399%	1.110%	1.980%
$\varepsilon^{MEAN}_{DH\ XPA}$	0.001%	0.012%	0.089%	0.129%	0.378%	0.669%
$\varepsilon^{MAX}_{DH\ KS}$	0.000%	0.007%	0.059%	0.096%	0.439%	0.994%
$\varepsilon^{MEAN}_{DH\ KS}$	0.000%	0.002%	0.012%	0.018%	0.094%	0.212%
$\varepsilon^{MAX}_{DH\ REITER}$	0.000%	0.003%	0.140%	0.376%	4.301%	9.591%
$\varepsilon^{MEAN}_{DH\ REITER}$	0.000%	0.002%	0.102%	0.264%	3.317%	7.407%